



# Fox Valley Math League

Menasha High School

February 28, 2011

Meet #5

Score:

Mathematician: \_\_\_\_\_ School Name: \_\_\_\_\_ Team \_\_\_\_\_

## EVENT 1

NO Calculators – Answers must be exact and completely simplified.

1. A number of children are standing equally spaced around the edge of a large circle and are numbered consecutively clockwise. The 14<sup>th</sup> child is directly opposite the 34<sup>th</sup> child. How many children are there?

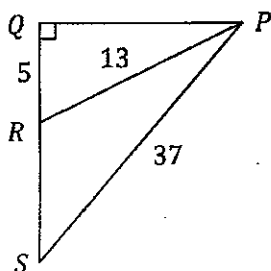
1. \_\_\_\_\_

2. Ms. Daniels recorded the test scores of her Algebra class of 25 students. The class average on the test was 72. Unfortunately, Samantha's test score of 86 was incorrectly recorded as 36. What is the correct test average?

2. \_\_\_\_\_

3. Given the diagram below, determine the perimeter of  $\triangle PQS$ .

3. \_\_\_\_\_



4. Suppose a function  $f$  is such that  $f\left(\frac{1}{x}\right) - 3f(x) = x$  where  $x \neq 0$ . Find  $f(2)$ .

4. \_\_\_\_\_



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## EVENT 2

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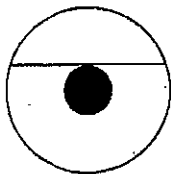
1. Two positive integers are chosen at random. What is the probability that their product is odd? 1. \_\_\_\_\_

2. What is the sum of the digits of the integer solution to: 3. \_\_\_\_\_

$$\sqrt{14 + \sqrt{27 - \sqrt{x - 1}}} = 4$$

3. The sum of the sixth and ninth terms of an arithmetic sequence is 20 and their product is 64. Find the tenth term if the first term is negative. 4. \_\_\_\_\_

4. The circles in the figure below are concentric. The chord is tangent to the inner circle and has a length of 12. What is the area of the non-shaded region? 5. \_\_\_\_\_





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## EVENT 3

**NO Calculators – Answers must be exact and completely simplified.**

1. Find all solutions for  $|2x - 6| = 4 - 3x$ .

1. \_\_\_\_\_

2. How many 5-digit numbers with all digits nonzero and no digit repeated are divisible by 25?

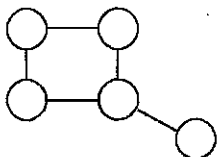
2. \_\_\_\_\_

3. Driving between two towns at 60 mph instead of 50 mph saves 5 minutes. What is the distance in miles between the two towns?

3. \_\_\_\_\_

4. Five circles are drawn on a piece of paper and connected as shown. Each circle must be colored red, blue or green. Two circles connected by a line segment may not be the same color. How many different ways are there to color the circles?

4. \_\_\_\_\_





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## EVENT 4

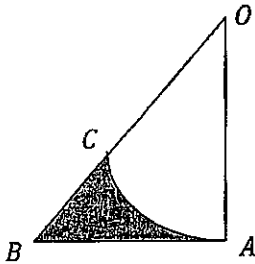
Calculators Allowed – All answers must be in exact simplified form or rounded to the nearest hundredth.

1. Find the linear function  $f(x)$  that satisfies both of the following conditions: 1. \_\_\_\_\_

$$f(1) = 3$$

$$f(x) = f(x + 1) - 2$$

2. In the figure,  $AC$  is an arc of circle with center  $O$ . If  $\overline{OA} \perp \overline{BA}$  and  $OA = BA = 14$ , what is the total area of the shaded region? 2. \_\_\_\_\_



3. Find the area of the region enclosed by the intersection of the graphs: 3. \_\_\_\_\_

$$y \leq -|x + 1| + 4 \quad \text{and} \quad y \geq |x + 1|$$

4. Inside a cube with edge length "a" there are 9 non-overlapping spheres with the same radius  $r$ . One sphere is concentric with the cube and touches the 8 other spheres, each of which touches 3 faces of the cube (is pushed into a corner). Express  $r$  in terms of  $a$ . 4. \_\_\_\_\_



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## TEAM EVENT

**Calculators Allowed – All answers must be in exact simplified form or rounded to the nearest hundredth.**

1. Allie's income is five-eighths that of Basil's. Allie's expenses are one-half those of Basil's, and Allie saves 40 percent of her income. What percentage of his income does Basil save? 1. \_\_\_\_\_

2. Two ships leave port at 9 AM. One travels at a bearing of  $53^\circ$  West of North at 10 mph. The other ship leaves at a bearing of  $67^\circ$  West of South at 15 mph. How far apart are they at noon that day (Round to the nearest hundredth)? 2. \_\_\_\_\_

3. The height of the water in a cylindrical tank lying on its side is 10 inches when the tank is half full. What is the height of the water when the tank is a quarter full? 3. \_\_\_\_\_

4. A point P is located in the interior of a rectangle such that the distance of P from a corner of the rectangle is 5 meters, from the opposite corner 14 meters, and from a third corner 10 meters. What is the distance P (in meters) from the fourth corner? 4. \_\_\_\_\_

5. Solve for  $x$  over the complex numbers: 5. \_\_\_\_\_

$$x^2 - (3 + i)x + (2 + 2i) = 0$$

6. Find the limit:  $\lim_{n \rightarrow 0} \frac{(5+n)^2 - 5^2}{n}$  6. \_\_\_\_\_



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### EVENT 1

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1. A number of children are standing equally spaced around the edge of a large circle and are numbered consecutively clockwise. The 14<sup>th</sup> child is directly opposite the 34<sup>th</sup> child. How many children are there?

1. 40

$$(34 - 14) \cdot 2 = 20 \cdot 2 = 40$$

2. Ms. Daniels recorded the test scores of her Algebra class of 25 students. The class average on the test was 72. Unfortunately, Samantha's test score of 86 was incorrectly recorded as 36. What is the correct test average?

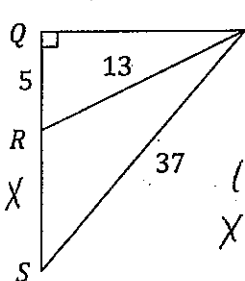
2. 74

$$\begin{array}{r} 36 \rightarrow 86 \\ +50 \end{array}$$

$$\frac{\sum}{25} = 72 \rightarrow \sum = 1800 + 50 = 1850 \quad \bar{X} = \frac{1850}{25} = 74$$

3. Given the diagram below, determine the perimeter of  $\triangle PQS$ .

3. 84



$$\begin{aligned} (QP)^2 + 5^2 &= 13^2 \\ (QP)^2 &= 144 \\ QP &= 12 \\ (X+5)^2 + 12^2 &= 37^2 \\ X^2 + 10X + 25 + 144 &= 1369 \\ X^2 + 10X - 1200 &= 0 \\ (X+40)(X-30) &= 0 \\ X &= 30 \end{aligned}$$

$$\begin{aligned} P &= 12 + 35 + 37 \\ P &= 84 \end{aligned}$$

4. Suppose a function  $f$  is such that  $f\left(\frac{1}{x}\right) - 3f(x) = x$  where  $x \neq 0$ . Find  $f(2)$ .

4.  $-\frac{13}{16}$

$$\begin{aligned} f\left(\frac{1}{2}\right) - 3f(2) &= 2 \rightarrow f\left(\frac{1}{2}\right) = 2 + 3f(2) \\ f(2) - 3f\left(\frac{1}{2}\right) &= \frac{1}{2} \\ f(2) - 3(2 + 3f(2)) &= \frac{1}{2} \\ f(2) - 6 - 9f(2) &= \frac{1}{2} \\ -8f(2) - 6 &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} -8f(2) &= \frac{13}{2} \\ f(2) &= -\frac{13}{16} \end{aligned}$$



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### EVENT 2

NO Calculators – Answers must be exact and completely simplified.

1. Two positive integers are chosen at random. What is the probability that their product is odd?

	Odd	Even
Odd	Odd	Even
Even	Even	Even

1.  $\frac{1}{4}$

2. What is the sum of the digits of the integer solution to:

$$\sqrt{14 + \sqrt{27 - \sqrt{x-1}}} = 4$$

$$14 + \sqrt{27 - \sqrt{x-1}} = 16$$

$$\sqrt{27 - \sqrt{x-1}} = 2$$

$$27 - \sqrt{x-1} = 4$$

$$\sqrt{x-1} = 23$$

$$x-1 = 529$$

$$x = 530 \rightarrow 5+3+0 = 8$$

3.  $8$

3. The sum of the sixth and ninth terms of an arithmetic sequence is 20 and their product is 64. Find the tenth term if the first term is negative.

$x = 6^{\text{th}} \text{ term}$   
 $y = 9^{\text{th}} \text{ term}$   
 $x + y = 20$      $x(y) = 64$   
 $y = 20 - x$

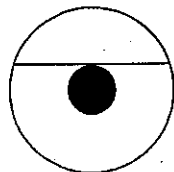
$x(20-x) = 64$   
 $20x - x^2 = 64$   
 $x^2 - 20x + 64 = 0$   
 $(x-4)(x-16) = 0$   
 $x = 4$  or  $x = 16$   
 $y = 16$      $y = 4$

$a_1 < 0 \rightarrow$  increasing sequence

$\frac{4}{6^{\text{th}}}$      $\frac{8}{7^{\text{th}}}$      $\frac{12}{8^{\text{th}}}$      $\frac{16}{9^{\text{th}}}$      $\frac{20}{10^{\text{th}}}$

4.  $20$

4. The circles in the figure below are concentric. The chord is tangent to the inner circle and has a length of 12. What is the area of the non-shaded region?



$r =$  radius of little  $\circ$   
 $R =$  radius of big  $\circ$

$r^2 + 6^2 = R^2$   
 $36 = R^2 - r^2$

$A_{\text{non-shaded}} = \pi R^2 - \pi r^2$   
 $= \pi (R^2 - r^2)$   
 $= 36\pi$

5.  $36\pi$



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### EVENT 3

NO Calculators – Answers must be exact and completely simplified.

1. Find all solutions for  $|2x - 6| = 4 - 3x$ .

$$2x - 6 = 4 - 3x \text{ or } 2x - 6 = 3x - 4$$

$$5x = 10 \qquad -x = 2$$

$$x = 2 \qquad x = -2$$

~~$x = 2$~~   
Extraneous!

1. -2

2. How many 5-digit numbers with all digits nonzero and no digit repeated are divisible by 25?

$$\frac{7}{7} \frac{6}{6} \frac{2}{5} \frac{5}{1} \frac{5}{1} \text{ OR } \frac{7}{7} \frac{6}{6} \frac{7}{5} \frac{5}{1} \frac{5}{1}$$

$$2(7P_3) = 2 \cdot 210 = 420$$

2. 420

3. Driving between two towns at 60 mph instead of 50 mph saves 5 minutes. What is the distance in miles between the two towns?

$$d = 60(t - \frac{5}{60}) \quad d = 50t$$

$$50t = 60t - 5 \quad d = 50(\frac{1}{2})$$

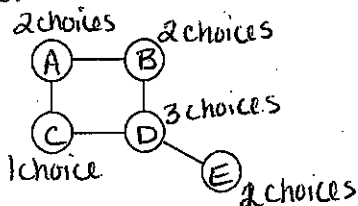
$$5 = 10t \quad d = 25 \text{ miles}$$

$$\frac{1}{2} = t$$

3. 25 miles

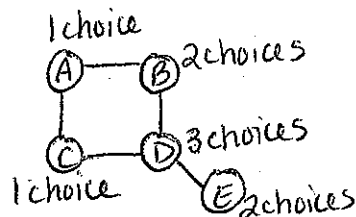
4. Five circles are drawn on a piece of paper and connected as shown. Each circle must be colored red, blue or green. Two circles connected by a line segment may not be the same color. How many different ways are there to color the circles?

Case 1: B+C are the same color.



$$2 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = 24$$

Case 2: B-C are different colors.



$$2 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 12$$

$$12 + 24 = \boxed{36}$$

4. 36



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### EVENT 4

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1. Find the linear function  $f(x)$  that satisfies both of the following conditions:

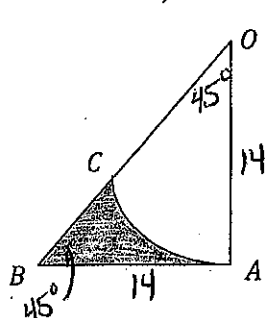
1.  $f(x) = 2x + 1$

$$\begin{aligned}
 f(1) &= 3 \\
 f(x) &= f(x+1) - 2 \\
 f(1) &= f(2) - 2 \\
 3 &= f(2) - 2 \\
 5 &= f(2)
 \end{aligned}$$

$m = \frac{2}{1}$   
 $3 = 2(1) + b$   
 $3 = 2 + b$   
 $1 = b$   
 $f(x) = 2x + 1$

2. In the figure,  $AC$  is an arc of circle with center  $O$ . If  $\overline{OA} \perp \overline{BA}$  and  $OA = BA = 14$ , what is the total area of the shaded region?

2.  $21.03$



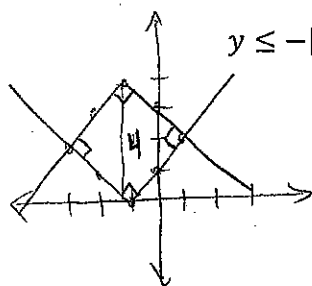
$$A_{\Delta} = \frac{14 \cdot 14}{2} = 98$$

$$\begin{aligned}
 A_{\text{sector}} &= \frac{45}{360} \cdot \pi \cdot 14^2 \\
 &= \frac{1}{8} \cdot \pi \cdot 196 \\
 &= 24.5\pi
 \end{aligned}$$

$$\begin{aligned}
 A_{\text{shaded}} &= A_{\Delta} - A_{\text{sector}} \\
 &= 98 - 24.5\pi \\
 &\approx 21.03
 \end{aligned}$$

3. Find the area of the region enclosed by the intersection of the graphs:

3.  $8$



$$y \leq -|x+1| + 4 \quad \text{and} \quad y \geq |x+1|$$

$$\frac{4}{\sqrt{2}} \cdot \frac{4}{\sqrt{2}} = \frac{16}{2} = 8$$

OR

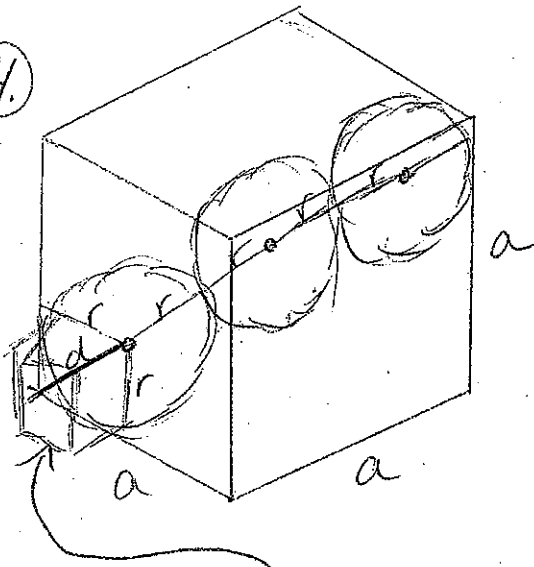
$$\frac{1}{2}d^2 = \frac{1}{2} \cdot 4^2 = 8$$

4. Inside a cube with edge length "a" there are 9 non-overlapping spheres with the same radius  $r$ . One sphere is concentric with the cube and touches the 8 other spheres, each of which touches 3 faces of the cube (is pushed into a corner). Express  $r$  in terms of  $a$ .

4.  $r = \frac{2\sqrt{3}-3}{2}a$  OR  $0.23a$

*Worked out solution on back.*

4.



Diagonal of cube =  $d$

$$d^2 = r^2 + r^2 + r^2$$

$$\sqrt{d^2} = \sqrt{3r^2}$$

$$d = \sqrt{3}r$$

Diagonal of Large Cube

$$4r + 2\sqrt{3}r = \sqrt{3}a$$

$$(4 + 2\sqrt{3})r = \sqrt{3}a$$

$$r = \frac{\sqrt{3}a}{4 + 2\sqrt{3}} \cdot \frac{4 - 2\sqrt{3}}{4 - 2\sqrt{3}}$$

$$= \frac{4\sqrt{3} - 6a}{4}$$

$$= \frac{2\sqrt{3} - 3a}{2}$$

$$\approx 0.23a$$



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## TEAM EVENT

Calculators Allowed – All answers must be in exact simplified form or rounded to the nearest hundredth.

- Allie's income is five-eighths that of Basil's. Allie's expenses are one-half those of Basil's, and Allie saves 40 percent of her income. What percentage of his income does Basil save? 1. 25%
- Two ships leave port at 9 AM. One travels at a bearing of  $53^\circ$  West of North at 10 mph. The other ship leaves at a bearing of  $67^\circ$  West of South at 15 mph. How far apart are they at noon that day (Round to the nearest hundredth)? 2.  $15\sqrt{7} \approx 39.69$
- The height of the water in a cylindrical tank lying on its side is 10 inches when the tank is half full. What is the height of the water when the tank is a quarter full? 3. 5.96
- A point P is located in the interior of a rectangle such that the distance of P from a corner of the rectangle is 5 meters, from the opposite corner 14 meters, and from a third corner 10 meters. What is the distance P (in meters) from the fourth corner? 4. 11 meters
- Solve for  $x$  over the complex numbers:  
$$x^2 - (3 + i)x + (2 + 2i) = 0$$
 5.  $1+i$  or  $2$
- Find the limit:  $\lim_{n \rightarrow 0} \frac{(5+n)^2 - 5^2}{n}$  6. 10

$$\textcircled{1} A_i = \frac{5}{8} B_i \quad A_E = \frac{1}{2} B_E$$

$$\frac{8}{5} A_i = B_i \quad 2A_E = B_E$$

$$\frac{A_i - A_E}{A_i} = \frac{2}{5} \rightarrow A_i - A_E = \frac{2}{5} A_i$$

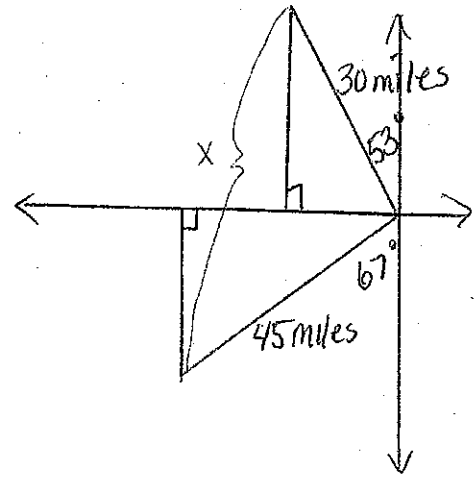
$$\frac{3}{5} A_i = A_E$$

$$\% \text{ saved} = \frac{B_i - B_E}{B_i} = \frac{\frac{8}{5} A_i - 2A_E}{\frac{8}{5} A_i}$$

$$= \frac{\frac{8}{5} A_i - 2(\frac{3}{5} A_i)}{\frac{8}{5} A_i}$$

$$= \frac{\frac{2}{5} A_i}{\frac{8}{5} A_i} = \frac{2}{8} = \frac{1}{4} = 25\%$$

②



$$x^2 = 30^2 + 45^2 - 2(30)(45)(\cos 60)$$

$$x^2 = 2925 + 7350$$

$$x^2 = 15775$$

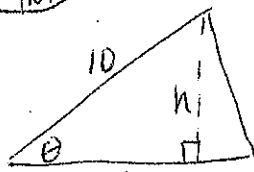
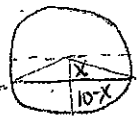
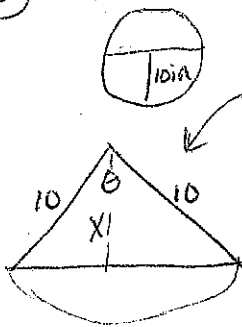
$$x = \sqrt{15775} \approx 39.69$$

$$x = 15\sqrt{7}$$

③

1/2 full

1/4 full



$$\sin \theta = \frac{h}{10}$$

$$10 \sin \theta = h$$

$$A_\theta = \frac{\theta}{2\pi} \quad A_\theta = \frac{10 \cdot 10 \sin \theta}{2}$$

$$50\theta - 50 \sin \theta = 25\pi$$

$$50(\theta - \sin \theta) = 25\pi$$

$$\theta - \sin \theta = \frac{\pi}{2}$$

$$\theta \approx 2.3098815$$

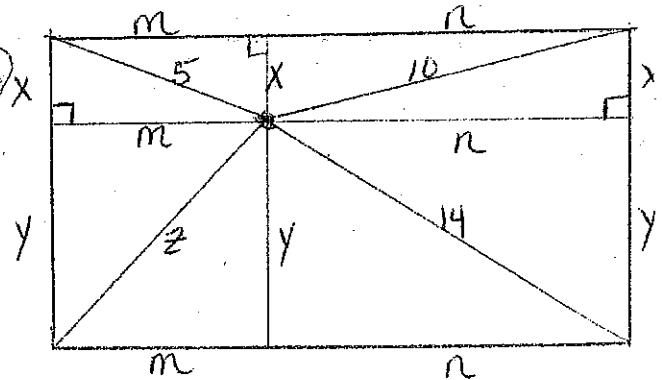
$$\cos \frac{\theta}{2} = \frac{x}{10}$$

$$10 \cos \frac{\theta}{2} = x$$

$$h = 10 - 10 \cos \frac{\theta}{2}$$

$$\approx 5.96$$

④



$$x^2 + m^2 = 5^2$$

$$x^2 = 5^2 - m^2$$

$$z^2 + 10^2 = 5^2 + 14^2$$

$$z^2 = 5^2 + 14^2 - 10^2$$

$$x^2 + n^2 = 10^2 \rightarrow n^2 = 10^2 - x^2$$

$$y^2 + n^2 = 14^2 \rightarrow n^2 = 14^2 - y^2$$

$$10^2 - x^2 = 14^2 - y^2$$

$$-x^2 = 14^2 - 10^2 - y^2$$

$$x^2 = y^2 + 10^2 - 14^2$$

$$5^2 - m^2 = y^2 + 10^2 - 14^2$$

$$14^2 + 5^2 - 10^2 - y^2 = m^2$$

$$z^2 = m^2 + y^2$$

$$z = \sqrt{5^2 + 14^2 - 10^2}$$

$$z = \sqrt{25 + 196 - 100}$$

$$z = 11$$

$$(5) X^2 - (3+i)X + (2+2i) = 0$$

$$X = \frac{3+i \pm \sqrt{-(3+i) \cdot -(3+i) - 4(2+2i)}}{2}$$

$$X = \frac{3+i \pm \sqrt{9+6i+i^2-8-8i}}{2}$$

$$X = \frac{3+i \pm \sqrt{i^2-2i+1}}{2}$$

$$X = \frac{3+i \pm \sqrt{(i-1)^2}}{2}$$

$$X = \frac{3+i \pm (i-1)}{2}$$

$$X = \frac{3+i+i-1}{2} \text{ OR } \frac{3+i-(i-1)}{2}$$

$$X = \frac{2+2i}{2} \text{ OR } \frac{4}{2}$$

$$X = 1+i \text{ OR } 2$$

$$(6) \lim_{n \rightarrow 0} \frac{(5+n)^2 - 5^2}{n} = \lim_{n \rightarrow 0} \frac{25+10n+n^2-25}{n}$$

$$= \lim_{n \rightarrow 0} \frac{n^2+10n}{n}$$

$$= \lim_{n \rightarrow 0} n+10$$

$$= 10$$